A Christofides-based approach to the travelling salesman problem in the unit cube

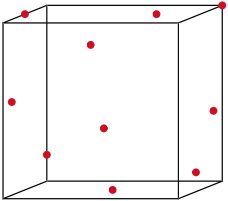
Alexey Gordeev

Umeå University, Sweden

August 28, 2025

TSP in the unit cube

find *Hamiltonian cycle* on $X \subseteq [0,1]^k$ with min. $\sum |e|^m$



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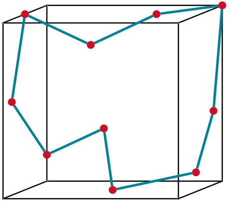

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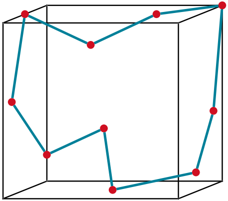

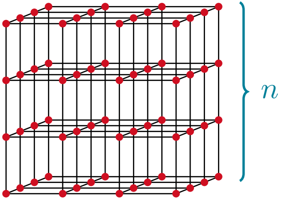

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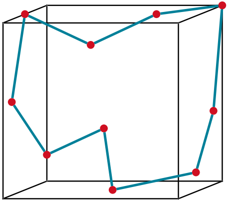

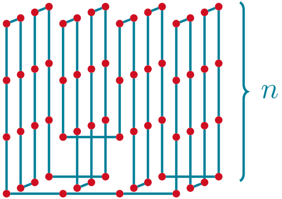

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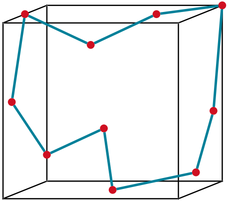

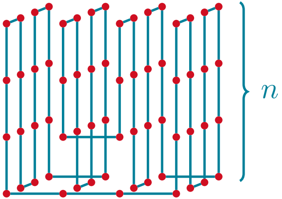

$$\sum |e|^m \approx \frac{n^k}{n^m} \xrightarrow{n\to\infty} \begin{cases} \infty & \text{if } k > m, \\ 0 & \text{if } k < m, \\ \mathbf{1} & \textbf{if } \mathbf{k = m}. \end{cases}$$

Bollobás–Meir conjecture

$\forall$ finite $X \subseteq [0,1]^k$ $\exists$ Ham. cycle H on X: $(\sum_H |e|^k)^{1/k} \leq \mathbf{s_k^{HC}}$

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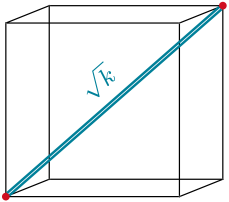

◇ $2^{1/k}\sqrt{k} \leq \mathbf{s_k^{HC}} \leq 9 \cdot (\frac{2}{3})^{1/k}\sqrt{k}$ **Bollobás–Meir 93**

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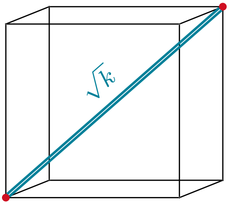

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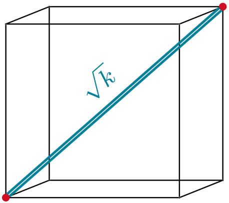

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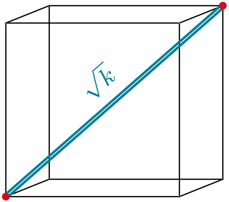

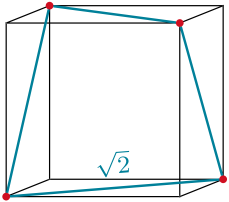

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Bollobás–Meir conjecture (upd. Balogh–Clemen–Dumitrescu 24)

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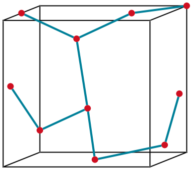
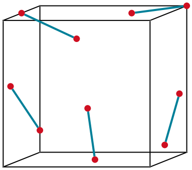
Results

- $\mathbf{s_k^{ST}}$ and $\mathbf{s_k^{PM}}$: analog. $\mathbf{s_k^{HC}}$ for *spanning trees* and *perfect matchings*





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$$\mathbf{s_k^{ST}} \leq \sqrt{\mathbf{5k}} \text{ or } \sqrt{k}(1 + o_k(1))$$

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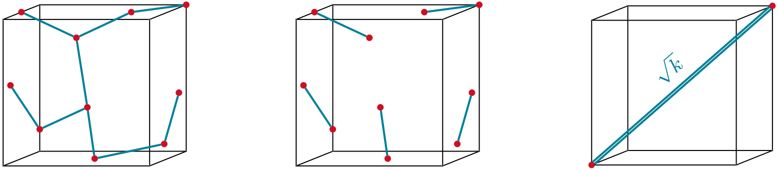

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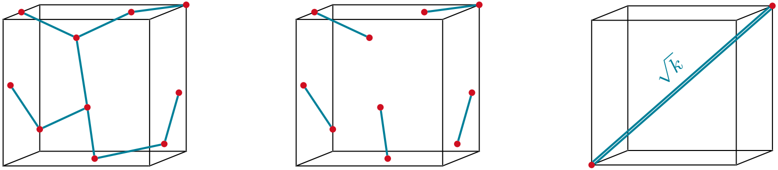

G 25+

$\mathbf{s_k^{PM}} \leq \mathbf{2^{1/k}\sqrt{2k}}$, $\quad \sqrt{\mathbf{5}} \Rightarrow \mathbf{1.823}$, $\quad \mathbf{6.709} \Rightarrow \mathbf{5.059}$, $\quad \mathbf{2.91} \Rightarrow \mathbf{2.65}$

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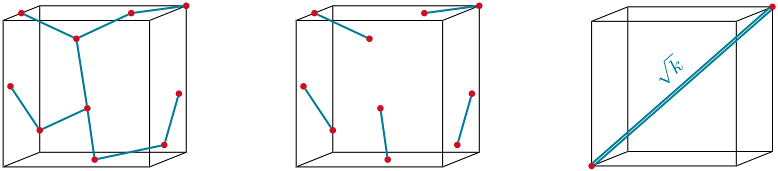

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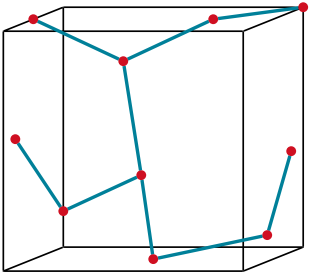
Tools: ball packing + cycle approximation

- min. spanning tree $\mathbf{T}$



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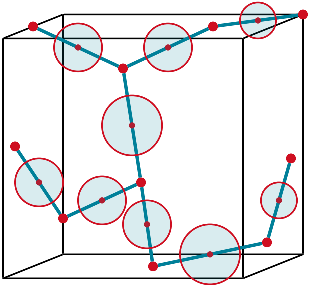
- min. spanning tree $\mathbf{T} \rightarrow \frac{|e|}{4}$-radius *ball packing* $\xrightarrow{\text{volume bound}} \mathbf{s_k^{ST}} \leq \sqrt{5k}$



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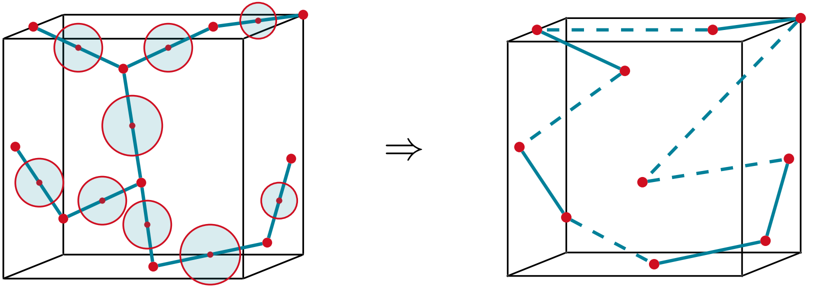
- min. spanning tree $\mathbf{T} \to \frac{|e|}{4}$-radius *ball packing* $\xrightarrow{\text{volume bound}} \mathbf{s_k^{ST}} \leq \sqrt{5k}$
- $\mathbf{T} \Rightarrow$ Hamiltonian cycle $\mathbf{H}$: $\mathbf{s_k^{HC}} \leq (\frac{2}{3})^{1/k} \cdot \mathbf{3} \cdot \mathbf{s_k^{ST}} \leq 6.709 \cdot (\frac{2}{3})^{1/k}\sqrt{k}$



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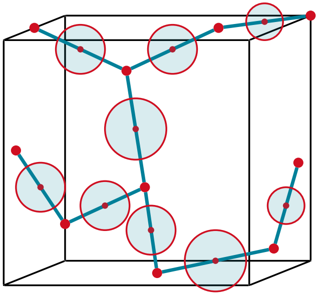

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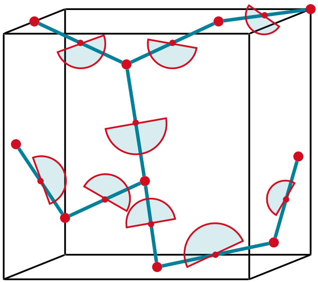

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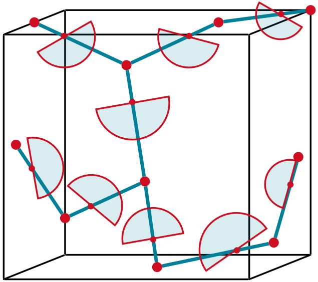

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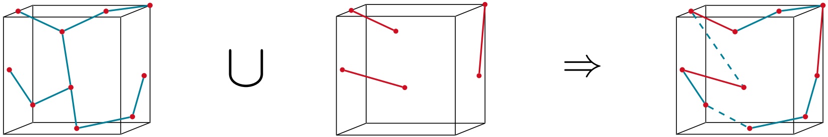

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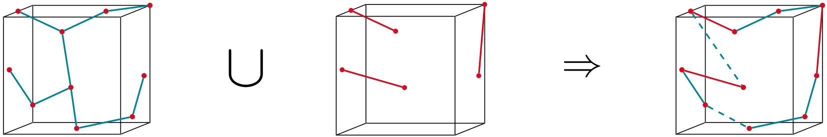

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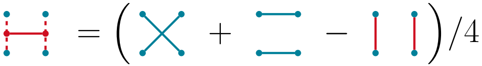

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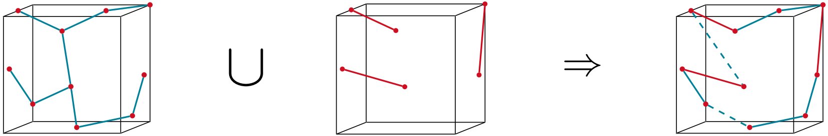

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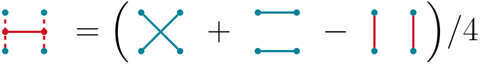

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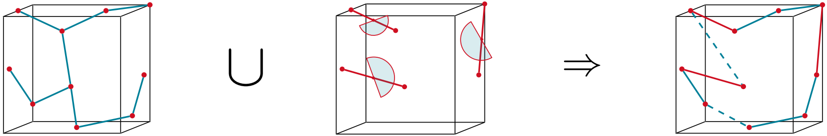

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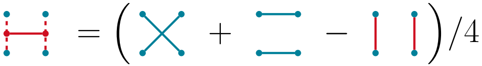

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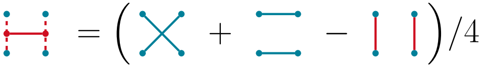
◇ $\frac{|e|}{2\sqrt{2}}$-*radius packing*: $\quad \mathbf{s_k^{PM}} \leq \mathbf{2^{1/k}\sqrt{2k}}, \quad \mathbf{s_k^{HC}} \leq \mathbf{5.059} \cdot (1.28)^{1/k}\sqrt{k}$

Ball packing for Hamiltonian cycles



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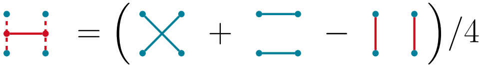

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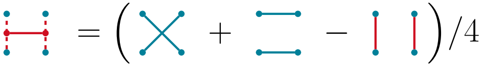

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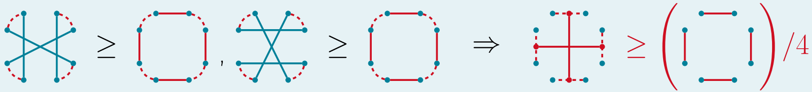

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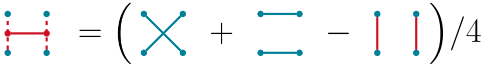

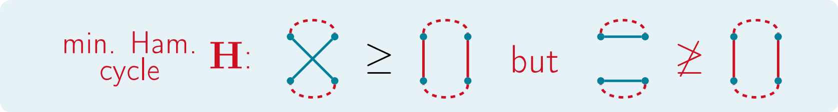

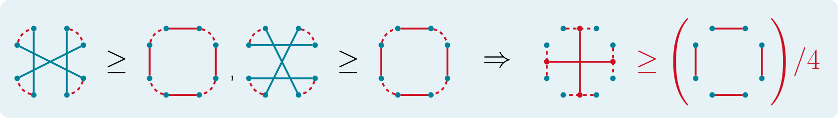

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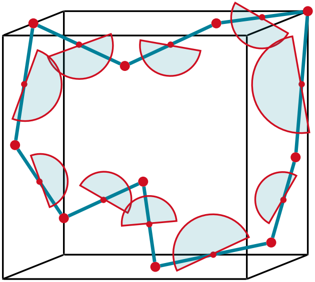

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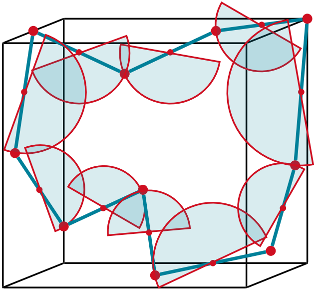

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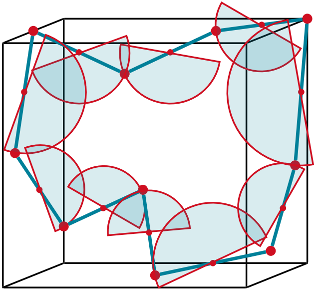
- small/large edges separately: $2^{1/k}\sqrt{k} \leq \mathbf{s_k^{HC}} \leq \mathbf{(2+o_k(1))^{1/k}}\sqrt{k}$



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$$\mathbf{s_k^{HC}} = 2^{1/k}\sqrt{k} \text{ for } k \neq 3, \quad \mathbf{s_3^{HC}} = 2^{7/6}$$

- $\frac{|e|}{2\sqrt{2}}$-radius **3-fold** packing: $\mathbf{s_k^{HC}} \leq \mathbf{6^{1/k}\sqrt{2k}}$
- *spherical codes* → $\frac{|\mathbf{e}|}{\mathbf{2}}$**-rad.** $\mathbf{3(k+1)}$-fold: $\mathbf{s_k^{HC}} \leq (\mathbf{6(k+1)})^{\mathbf{1/k}}\sqrt{k}$
- small/large edges separately: $2^{1/k}\sqrt{k} \leq \mathbf{s_k^{HC}} \leq (\mathbf{2 + o_k(1)})^{\mathbf{1/k}}\sqrt{k}$