Near Triple Arrays

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Based on joint work with Klas Markström and Lars-Daniel Öhman

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Triple arrays (TA)

$(r \times c, v)$-triple array

$r \times c$ table, v symbols each used e times, no repetitions in rows/columns,

$$|\text{row} \cap \text{row}| = \lambda_{rr}, \ |\text{column} \cap \text{column}| = \lambda_{cc}, \ |\text{row} \cap \text{column}| = \lambda_{rc}$$

1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	1	10	11	8	12	9	3
12	10	11	9	7	1	4	2	6

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1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	**1**	10	11	8	12	9	3
12	10	11	9	7	**1**	4	2	6

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1	2	**3**	4	**5**	6	**7**	**8**	**9**
2	3	4	5	6	10	8	11	12
5	**7**	**1**	10	11	**8**	12	**9**	**3**
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12	10	11	9	**7**	1	4	2	6

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2	**3**	**4**	5	6	10	8	**11**	12
5	7	1	10	11	8	12	9	3
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- Statistically optimal as experimental designs
- *Latin square* = $(n \times n, n)$-TA, *Youden rectangle* = $(r \times n, n)$-TA

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- ◇ Statistically optimal as experimental designs
- ◇ *Latin square* = $(n \times n, n)$-TA, *Youden rectangle* = $(r \times n, n)$-TA

$$e = \lambda_{rc} = \frac{rc}{v}, \quad \lambda_{rr} = \frac{c(e-1)}{r-1}, \quad \lambda_{cc} = \frac{r(e-1)}{c-1}$$

- *admissible* $(r \times c, v)$: $e, \lambda_{rc}, \lambda_{rr}, \lambda_{cc} \in \mathbb{Z}, \quad \max(r, c) \leq v \leq rc$

Near triple arrays (NTA)

$(r \times c, v)$-near triple array **G.–Markström–Öhman 25**

each symbol used $\lfloor e \rfloor$ or $\lceil e \rceil$ times, no repetitions in rows/columns,

$|\text{row} \cap \text{row}| = \lfloor \lambda_{rr} \rfloor$ or $\lceil \lambda_{rr} \rceil$, $|\text{column} \cap \text{column}| = \lfloor \lambda_{cc} \rfloor$ or $\lceil \lambda_{cc} \rceil$,

$|\text{row} \cap \text{column}| = \lfloor \lambda_{rc} \rfloor$ or $\lceil \lambda_{rc} \rceil$

1	2	3	4	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	1	4	2

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2	**3**	4	7	8	9
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1	**2**	3	**4**	5	**6**
2	3	4	**7**	**8**	**9**
5	1	**7**	**9**	6	**8**
7	8	**6**	**1**	**4**	**2**

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1	2	3	**4**	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	**1**	4	2

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1	2	3	4	5	6
2	3	4	7	**8**	9
5	1	7	9	**6**	8
7	**8**	**6**	1	**4**	2

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1	2	3	4	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	1	4	2

- No admissibility conditions
- NTA = TA for TA-admissible $(r \times c, v)$

Component designs

1	2	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA

Component designs

1	2	3	4	5	6
2	3	**1**	7	8	9
4	10	8	6	9	**1**
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA

1 

Column design

Component designs

1	**2**	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	**2**	5
10	9	5	3	4	7

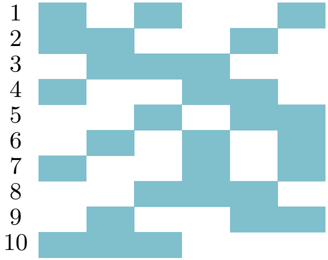
$(5 \times 6, 10)$-TA

Column design

Component designs

1	2	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA

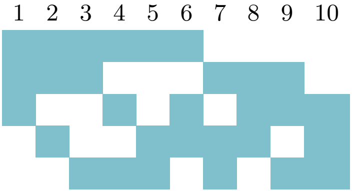


Column design

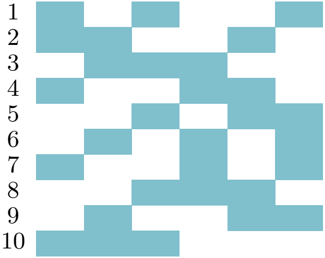
Component designs

1	2	3	4	5	6
2	3	1	7	8	9
4	10	8	6	9	1
7	6	10	8	2	5
10	9	5	3	4	7

$(5 \times 6, 10)$-TA



Row design

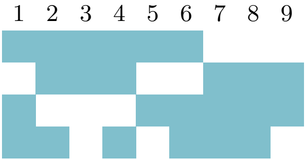


Column design

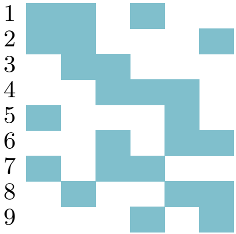
Component designs

1	2	3	4	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	1	4	2

$(4 \times 6, 9)$-NTA



Row design

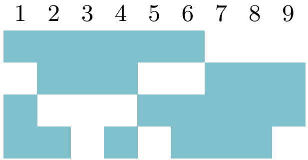


Column design

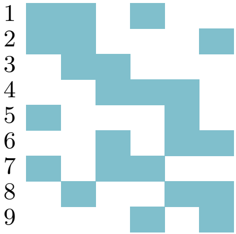
Component designs

1	2	3	4	5	6
2	3	4	7	8	9
5	1	7	9	6	8
7	8	6	1	4	2

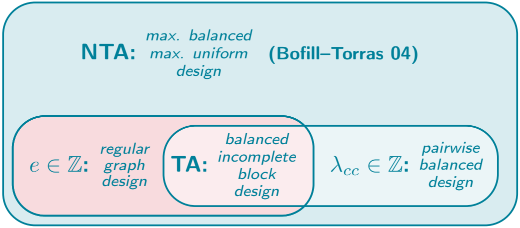
$(4 \times 6, 9)$-NTA



Row design



Column design



Existence of NTA

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3×3	+	+	+	−	+	+	+																																									
3×4		+	+	−	+	+	+	+	+	+																																						
3×5			+	+	+	−	+	+	+	+	+	+	+																																			
3×6				+	+	+	+	+	+	+	+	+	+	+	+	+																																
3×7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																													
3×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
3×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																							
3×10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
3×11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																	
3×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
3×13											+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+											
3×14												+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3×15													+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4×4		+	+	+	+	+	−	+	+	+	+	+	+	+																																		
4×5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+																														
4×6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
4×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																						
4×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																		
4×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
4×10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4×11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5×5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																									
5×6				+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
5×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+															
5×8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5×9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5×10								+	+	−	−	+	+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
6×7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6×8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7×7					+	+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

Existence of NTA

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3×3	+	+	+	−	+	+	+																																									
3×4		+	+	**−**	+	+	+	+	+	+																																						
3×5			+	+	+	−	+	+	+	+	+	+	+																																			
3×6				+	+	+	+	+	+	+	+	+	+	+	+	+																																
3×7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																													
3×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
3×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																							
3×10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
3×11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																	
3×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
3×13											+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+											
3×14												+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3×15													+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4×4		+	+	+	+	+	−	+	+	+	+	+	+	+																																		
4×5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+																														
4×6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
4×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																						
4×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																		
4×9							+	+	+	**+**	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
4×10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4×11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5×5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																									
5×6				+	+	−	+	**+**	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
5×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+															
5×8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5×9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5×10								+	+	−	−	+	+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
6×7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6×8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7×7					+	+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

Existence of NTA

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3×3	+	+	+	−	+	+	+																																									
3×4		+	+	−	+	+	+	+	+	+																																						
3×5			+	+	+	−	+	+	+	+	+	+	+																																			
3×6				+	+	+	+	+	+	+	+	+	+	+	+	+																																
3×7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																													
3×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
3×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																							
3×10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
3×11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																	
3×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
3×13											+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+											
3×14												+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3×15													+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4×4		+	+	+	+	+	−	+	+	+	+	+	+	+																																		
4×5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+																														
4×6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																										
4×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																						
4×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																		
4×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
4×10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4×11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5×5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																									
5×6				+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+																				
5×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+															
5×8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5×9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5×10								+	+	−	−	+	+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+														
6×7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6×8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7×7					+	+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

- ◇ $(3\times c, v)$-NTA exist for all v when $c \geq 6$ **G.–Markström–Öhman 25**
- ◇ **Conjecture:** $(r\times c, v)$-NTA exist for all v when $c \geq r(r-1)$

Existence of NTA

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
3×3	+	+	+	−	+	+	+																	
3×4		+	+	−	+	+	+	+	+	+														
3×5			+	+	+	−	+	+	+	+	+	+	+											
3×6				+	+	+	+	+	+	+	+	+	+	+	+	+								
3×7					+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
3×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
3×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3×10								+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3×11									+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
3×13											+	+	+	+	+	+	+	+	+	+	+	+	+	+
3×14												+	+	+	+	+	+	+	+	+	+	+	+	+
3×15													+	+	+	+	+	+	+	+	+	+	+	+
4×4		+	+	+	+	+	−	+	+	+	+	+	+	+										
4×5			+	+	−	+	+	−	+	+	+	+	+	+	+	+	+	+						
4×6				+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
4×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
4×8						+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
4×9							+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
4×10								+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+
4×11									+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+
4×12										+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
5×5			+	+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	
5×6				+	+	−	+	+	+	+	+	+	−	+	+	+	+	+	+	+	+	+	+	+
5×7					+	+	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
5×8						+	+	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
5×9							+	+	−	−	+	+	−	+	+	+	+	+	+	+	+	+	+	+
5×10								+	+	−	−	+	+	+	−	−	+	+	+	+	+	+	+	+
6×6				+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×7					+	+	+	−	−	+	+	−	+	+	+	+	+	+	−	+	+	+	+	+
6×8						+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+
7×7					+	+	+	−	−	−	+	+	+	+	+	+	+	+	+	+	+	+	+	+

	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
3×3																								
3×4																								
3×5																								
3×6																								
3×7																								
3×8																								
3×9	+																							
3×10	+	+	+	+																				
3×11	+	+	+	+	+	+	+																	
3×12	+	+	+	+	+	+	+	+	+	+														
3×13	+	+	+	+	+	+	+	+	+	+	+	+	+											
3×14	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
3×15	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
4×4																								
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4×7	+	+																						
4×8	+	+	+	+	+	+																		
4×9	+	+	+	+	+	+	+	+	+	+														
4×10	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
4×11	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+						
4×12	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
5×5																								
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5×7	+	+	+	+	+	+	+	+	+															
5×8	+	+	+	+	+	+	+	+	+	+	+	+	+	+										
5×9	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+					
5×10	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
6×6	+	+	+	+	+	+	+	+	+	+														
6×7	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+								
6×8	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+		
7×7	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	

- ◇ $(3 \times c, v)$-NTA exist for all v when $c \geq 6$ **G.–Markström–Öhman 25**
- ◇ **Conjecture:** $(r \times c, v)$-NTA exist for all v when $c \geq r(r-1)$
- ◇ **Question:** arrays with $|\text{row} \cap \text{row}| \in [x, x+2]$, etc., exist for all r, c, v?

Connections with balanced grids

- *Balanced grid*: each pair of symbols appears together in μ rows+columns

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A. $v \geq r + c - 1$ for TA **Bailey–Heidtmann 94, Bagchi 98, MPWY 05**

B. $v \leq r + c - 1$ for BG **MPWY 05**

C. TA $\Leftrightarrow$ BG when $v = r + c - 1$ **MPWY 05 + McSorley 05**

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- *Near balanced grid*: ... in $\lfloor \mu \rfloor$ or $\lceil \mu \rceil$ rows+columns

$(r \times c, v)$**-array:**

$$\# \begin{pmatrix} \text{pair of rows/columns} \\ \text{both contain same} \\ \text{pair of symbols} \end{pmatrix} \begin{array}{l} \geq S_{\mathrm{NTA}} = S_{\mathrm{NTA}}(r, c, v) \\ \geq S_{\mathrm{NBG}} = S_{\mathrm{NBG}}(r, c, v) \end{array}$$

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$(r \times c, v)$**-array:** $\# \begin{pmatrix} \text{pair of rows/columns} \\ \text{both contain same} \\ \text{pair of symbols} \end{pmatrix} \begin{matrix} \geq S_{\mathrm{NTA}} = S_{\mathrm{NTA}}(r, c, v) \\ \geq S_{\mathrm{NBG}} = S_{\mathrm{NBG}}(r, c, v) \end{matrix}$

Theorem **G.–Markström–Öhman 25**

$S_{\mathrm{NTA}} < S_{\mathrm{NBG}} \Rightarrow$ **no NTA** $S_{\mathrm{NTA}} > S_{\mathrm{NBG}} \Rightarrow$ **no NBG**

$S_{\mathrm{NTA}} = S_{\mathrm{NBG}} \Rightarrow$ **NTA = NBG**

◇ **Implies A, B, C**

Enumeration of TA

- *isotopism* = row + column + symbol permutations

	v	6	10	12	14	15	20
	$r \times c$	3×4	5×6	4×9	7×8	6×10	5×16
Total #		0	7	1	684782	270119	26804
$\lvert \mathrm{Aut} \rvert$	1				682054	263790	26714
	2				1266	5280	
	3		2	1	1277	260	90
	4		1		98	579	
	5					1	
	6		1		48	69	
	7				2		
	8				12	88	
	10					2	
	12		2		9	17	
	16					11	
	18					1	
	20					4	
	21				8		
	24				7	9	
	36					2	
	48					4	
	60		1				
	120					1	
	168				1		
	720					1	

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	4		1		98	579	
	5					1	
	6		1		48	69	
	7				2		
	8				12	88	
	10					2	
	12		2		9	17	
	16					11	
	18					1	
	20					4	
	21				8		
	24				7	9	
	36					2	
	48					4	
	60		1				
	120					**1**	
	168				**1**		
	720					**1**	

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	5	6	7	8	9
1	0	3	2	5	4	10	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	7	14	5	3	0
11	7	10	6	9	13	3	1	14	4
12	8	9	13	10	6	5	14	1	2

- "*intercalate*-dense"

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	5	6	7	8	9
1	0	**3**	2	5	4	10	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	7	14	5	3	0
11	7	10	6	9	13	3	1	14	4
12	8	9	13	10	6	5	14	1	2

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0	1	2	3	4	5	6	7	8	9
1	0	**3**	2	5	4	10	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	7	14	5	3	0
11	7	10	6	9	13	**3**	1	14	4
12	8	9	13	10	6	5	14	1	2

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$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	5	6	7	8	9
1	0	**3**	2	5	4	**10**	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	7	14	5	3	0
11	7	**10**	6	9	13	**3**	1	14	4
12	8	9	13	10	6	5	14	1	2

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$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	**5**	6	7	8	9
1	0	**3**	2	5	4	**10**	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	7	14	5	3	0
11	7	**10**	6	9	13	**3**	1	14	4
12	8	9	13	10	6	5	14	1	2

- "*intercalate*-dense"

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	**5**	6	7	8	9
1	0	**3**	2	5	4	**10**	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	7	14	**5**	3	0
11	7	**10**	6	9	13	**3**	1	14	4
12	8	9	13	10	6	5	14	1	2

- "*intercalate*-dense"

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	**5**	6	**7**	8	9
1	0	**3**	2	5	4	**10**	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	**7**	14	**5**	3	0
11	7	**10**	6	9	13	**3**	1	14	4
12	8	9	13	10	6	5	14	1	2

- "*intercalate*-dense"

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 720$

0	1	2	3	4	**5**	6	**7**	8	9
1	0	**3**	2	5	4	**10**	11	12	13
6	10	7	11	8	12	0	2	4	14
9	13	12	8	11	**7**	14	**5**	3	0
11	7	**10**	6	9	13	**3**	1	14	4
12	8	9	13	10	6	5	14	1	2

- "*intercalate*-dense"
  - part of infinite series of intercalate-dense TA **Nilson 22**

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 120$

0	1	2	3	4	5	6	7	8	9
1	0	3	4	5	2	10	11	12	13
6	11	0	8	14	13	9	12	3	2
8	13	14	11	0	6	5	4	7	10
10	9	7	14	12	1	11	2	5	8
12	7	10	1	9	14	3	6	13	4

- **intercalate partition**

$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 120$

0	**1**	2	3	4	5	6	7	8	9
1	**0**	3	4	5	2	10	11	12	13
6	11	**0**	8	**14**	13	9	12	3	2
8	13	**14**	11	**0**	6	5	4	7	10
10	9	7	**14**	12	**1**	11	2	5	8
12	7	10	**1**	9	**14**	3	6	13	4

- **intercalate partition**

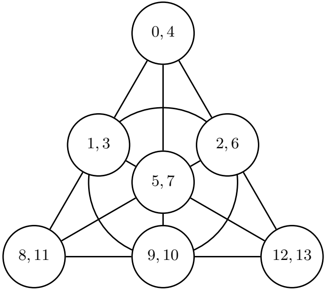
$(6 \times 10, 15)$-TA with $|\,\mathrm{Aut}\,| = 120$

0	1	**2**	3	4	5	6	**7**	8	9
1	0	3	4	5	**2**	10	11	12	**13**
6	11	0	8	14	**13**	9	12	3	**2**
8	**13**	14	11	0	6	5	4	**7**	10
10	9	**7**	14	12	1	11	**2**	5	8
12	**7**	10	1	9	14	3	6	**13**	4

- **intercalate partition**

$(7 \times 8, 14)$-TA with $|\mathrm{Aut}| = 168$

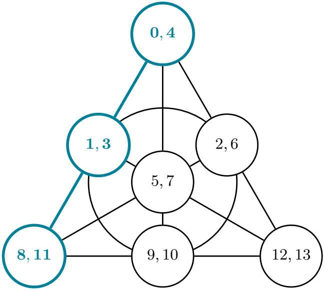
0	1	2	3	4	5	6	7
1	8	3	9	5	10	7	11
2	12	13	8	3	1	11	6
7	5	9	2	6	12	13	10
8	0	5	7	13	11	4	12
10	6	11	4	8	2	9	0
13	9	0	12	10	4	1	3



- line of *Fano plane* $\leftrightarrow$ four 3×2 *Latin subrectangles*

$(7 \times 8, 14)$-TA with $|\,\mathrm{Aut}\,| = 168$

0	**1**	2	**3**	**4**	5	6	7
1	**8**	**3**	9	5	10	7	**11**
2	12	13	**8**	**3**	**1**	**11**	6
7	5	9	2	6	12	13	10
8	**0**	5	7	13	**11**	**4**	12
10	6	**11**	**4**	**8**	2	9	**0**
13	9	**0**	12	10	**4**	**1**	**3**



- line of *Fano plane* $\leftrightarrow$ four 3×2 *Latin subrectangles*