Combinatorial Nullstellensatz and the Erdős box problem

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Combinatorial Nullstellensatz

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- $x_1^{d_1} \ldots x_r^{d_r}$ is a *monomial of f* if its coefficient in f is non-zero

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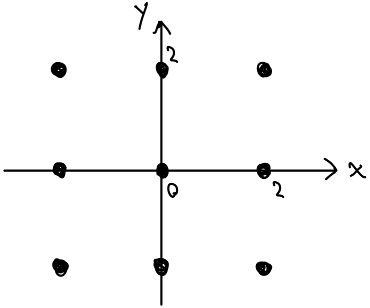
$f(x, y) = \cdots + cx^2y^2$

$A_1 = A_2 = \{-2, 0, 2\}$



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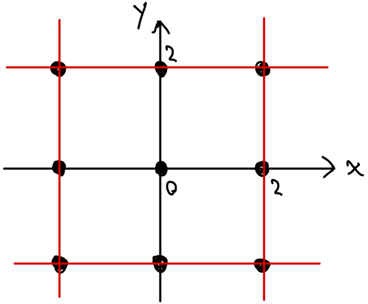

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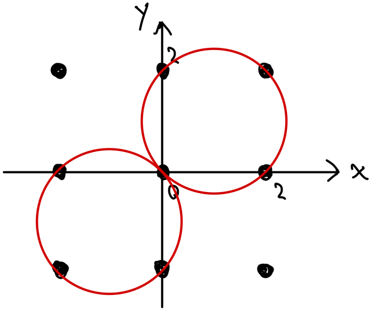

Example: a proof of Cauchy-Davenport theorem

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- ◇ *Example*: $f(x, y) = x^{100} + xy + y^{100}$
- ◇ **Schauz, 2008**: even more general theorem
- ◇ in practically all known applications the degree condition is sufficient

Turán numbers

- G — graph; *Turán number* (or *extremal number*) $\mathrm{ex}(n, G)$ — maximum number of edges in a graph on n vertices containing no copies of G

Theorem (Turán, 1941)

$\mathrm{ex}(n, K_{m+1}) = \left(1 - \frac{1}{m} + o(1)\right)\binom{n}{2}$.

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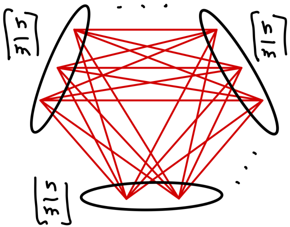

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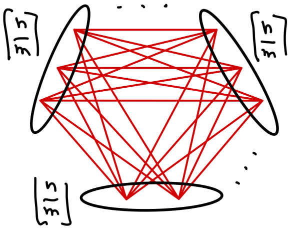

Theorem (Erdős–Stone, 1946)

$$\mathrm{ex}(n, G) = \left(1 - \frac{1}{\chi(G)-1} + o(1)\right)\binom{n}{2}.$$

◇ determines $\mathrm{ex}(n, G)$ asymptotically when G is not bipartite ($\chi(G) > 2$)

Hypergraph Turán numbers and r-partite r-graphs

- *hypergraph* $H = (V, E)$: V — set of vertices, $E \subset 2^V$ — set of edges
- *r-graph* (or *r-uniform hypergraph*): every edge contains r vertices
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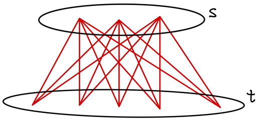

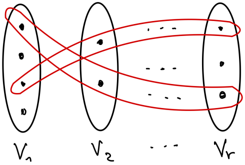

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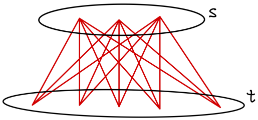
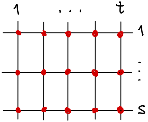
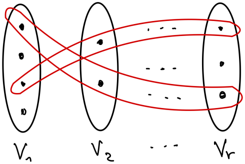
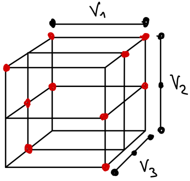
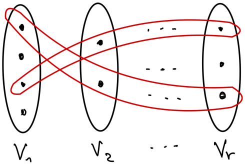


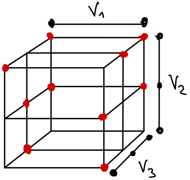






Turán numbers of complete r-partite r-graphs





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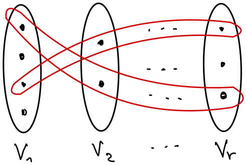

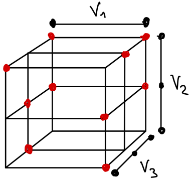

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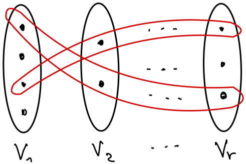

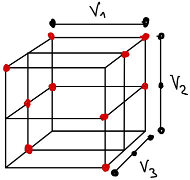

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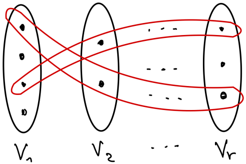

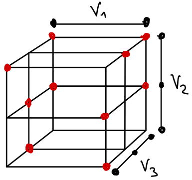

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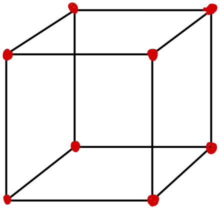
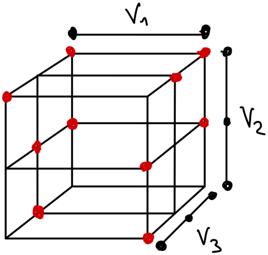
- $s_i \ge 2$ and $s_r \ge C(s_1, \dots, s_{r-1})$ **Ma–Yuan–Zhang, 2018**
  (for graphs: **Blagojević–Bukh–Karasev, 2013**, **Bukh, 2015**)
- $s_i \ge 2$ and $s_r > ((r-1)(s_1 \dots s_{r-1} - 1))!$ **Pohoata–Zakharov, 2021+**
  (for graphs: **Kollár–Rónyai–Szabó, 1996**, **Alon–Rónyai–Szabó, 1999**)

Erdős box problem





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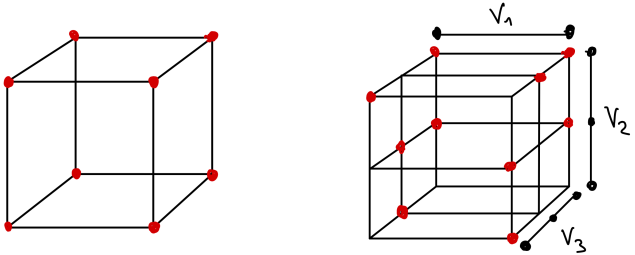

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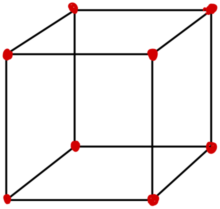

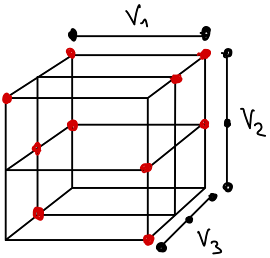

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- $\mathrm{ex}(n, K^{(3)}_{2,2,2}) = \Omega(n^{8/3})$ **Katz–Krop–Maggioni, 2002**

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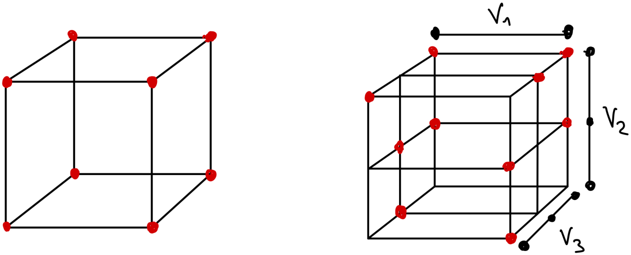

- $\mathrm{ex}(n, K^{(r)}_{2,\dots,2}) = O\left(n^{r-\frac{1}{2^{r-1}}}\right)$ for $r \geq 2$ **Erdős, 1964**
- $\mathrm{ex}(n, K_{2,2}) = \Theta(n^{3/2})$ **E.Klein, 1934**

◇ no matching lower bound for $r > 2$ is known

- $\mathrm{ex}(n, K^{(3)}_{2,2,2}) = \Omega(n^{8/3})$ **Katz–Krop–Maggioni, 2002**
- $\mathrm{ex}(n, K^{(r)}_{2,\dots,2}) = \Omega\left(n^{r-\lceil\frac{2^r-1}{r}\rceil^{-1}}\right)$ **Conlon–Pohoata–Zakharov, 2021**
  (improving on **Gunderson–Rödl–Sidorenko, 1999**)

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$r - \frac{1}{2^{r-1}}$	1.5	2.75	3.875	4.9375	5.96875
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  - proof is much more complicated

The framework

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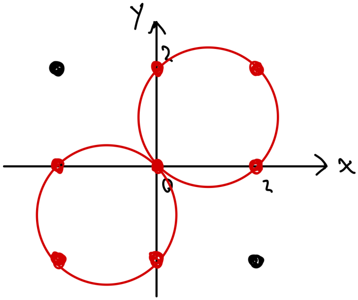

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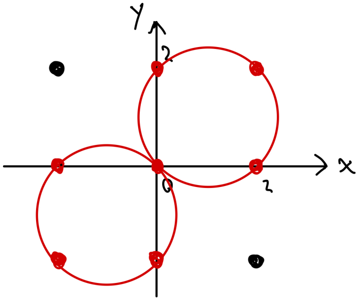

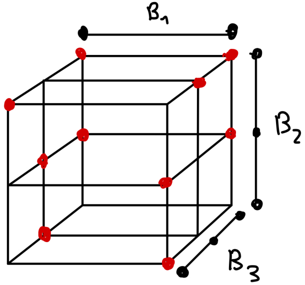

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If $x_1^{d_1} \ldots x_r^{d_r}$ is a maximal monomial of f, then for any $A_1, \ldots, A_r \subset \mathbb{F}$ with $|A_i| \geq d_i + 1$, $f(a_1, \ldots, a_r) \neq 0$ for some $a_i \in A_i$. In other words, f does not vanish on $A_1 \times \cdots \times A_r$.

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Key Lemma (general)

If for every $\pi \in \mathcal{S}_r$ there exists a maximal monomial of f which divides $x_{\pi_1}^{d_1} \dots x_{\pi_r}^{d_r}$, then for any $B_1, \dots, B_r \subset \mathbb{F}$, $H(f, B_1 \times \cdots \times B_r)$ is free of copies of $K^{(r)}_{d_1+1,\dots,d_r+1}$.

Schwartz–Zippel type corollary

Theorem (Erdős, 1964; for graphs: Kővári–Sós–Turán, 1954)

$\mathrm{ex}(n, K^{(r)}_{s_1,\dots,s_r}) = O\left(n^{r-\frac{1}{s_1\cdots s_{r-1}}}\right)$ for $s_1 \le \cdots \le s_r$.

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- *Example*: $|Z(x^n + xy + y^n, B_1\times B_2)| = O(n^{3/2})$ for any B_i, $|B_i| = n$

Construction

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Lemma (G., 2024)

Let $\mathbb{F}_{p^r}$ be the finite field of size p^r, $\mathbb{F}^*_{p^r} = \mathbb{F}_{p^r} \setminus \{0\}$, and

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2. $H(f, (\mathbb{F}^*_{p^r})^r)$ has $n = r(p^r - 1)$ vertices.
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- When is the bound from the corollary not optimal?
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  - ◇ Can other known constructions be "translated" into this language?

Corollary

If $x_1^{d_1} \dots x_r^{d_r}$ is a maximal monomial of f, $d = \max(d_1, \dots, d_r)$, then for any $B_1, \dots, B_r \subset \mathbb{F}$, $|B_i| = n$,

$$|Z(f, B_1 \times \cdots \times B_r)| = O\left(n^{r - \frac{1}{(d+1)^{r-1}}}\right).$$

- When is the bound from the corollary not optimal?
  - ◇ It is tight for $f(x,y) = xy + P(x) + Q(y)$, $\mathbb{F} = \mathbb{F}_{p^2}$
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  - ◇ **Rote, 2023**: How large can the set $Z(f, B_1 \times B_2)$ be for $f(x,y) = xy + P(x) + Q(y)$, $B_i \subset \mathbb{Z}$, $|B_i| = n$?

Further questions

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- Find more applications of Lasoń's Combinatorial Nullstellensatz