New Non-Extremal Triple Arrays

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Based on joint work with Lars-Daniel Öhman

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What is a triple array?

- An $(r \times c)$-array on v symbols with no repetitions in rows or columns
- every symbol appears e times
- every two rows have λ_{rr} common symbols
- every two columns have λ_{cc} common symbols
- every row and column have λ_{rc} common symbols

1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	1	10	11	8	12	9	3
12	10	11	9	7	1	4	2	6

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5	7	**1**	10	11	8	12	9	3
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Why triple arrays?

- ◇ Great experimental designs: efficient in eliminating the effects of two factors
- ◇ Have interesting combinatorial structure; generalize latin squares and Youden rectangles

What is a triple array?

- $(r \times c, rc)$-triple array

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

- $(n \times n, n)$-triple array: *latin square*

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

- $(n \times k, n)$-triple array: *Youden rectangle*

1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

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1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

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1	2	3	4	5
2	1	4	5	3
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1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

- Usually it is assumed that $\max(r, c) < v < rc$

What is a triple array?

◇ $(r \times c, rc)$-triple array

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24

◇ $(n \times n, n)$-triple array: *latin square*

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

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1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

- Usually it is assumed that $\max(r, c) < v < rc$

◇ $(r \times c, v)$ determine $e = \lambda_{rc} = \frac{rc}{v}$, $\lambda_{rr} = \frac{c(e-1)}{r-1}$, $\lambda_{cc} = \frac{r(e-1)}{c-1}$

- $(r \times c, v)$ are *admissible* if $e, \lambda_{rr}, \lambda_{cc}, \lambda_{rc}$ are integers, $\max(r, c) < v < rc$

Extremal triple arrays

- ◇ $v \geq r + c - 1$ **Bagchi, 1998; McSorley–Phillips–Wallis–Yucas, 2005**
- $(r \times c, v)$-TA is *extremal* if $v = r + c - 1$

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- *Construction*: **Agrawal, 1966**

  symmetric 2-design $\xrightarrow{\text{assignment problem}}$ extremal TA

		C_1	C_2	C_3	C_4	C_5	C_6
		1,2,4,7,10	2,3,6,9,10	1,3,5,8,10	3,4,6,7,8	2,4,5,8,9	1,5,6,7,9
R_1	1,2,3,4,5,6						
R_2	1,2,3,7,8,9						
R_3	1,4,6,8,9,10						
R_4	2,5,6,7,8,10						
R_5	3,4,5,7,9,10						

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		C_1 1,2,4,7,10	C_2 2,3,6,9,10	C_3 1,3,5,8,10	C_4 3,4,6,7,8	C_5 2,4,5,8,9	C_6 1,5,6,7,9
R_1	1,2,3,4,5,6	**1**, 2, 4	**2**, 3, 6	1, **3**, 5	3, **4**, 6	2, 4, **5**	1, 5, **6**
R_2	1,2,3,7,8,9	1, **2**, 7	2, **3**, 9	**1**, 3, 8	3, **7**, 8	2, **8**, 9	1, 7, **9**
R_3	1,4,6,8,9,10	1, **4**, 10	6, 9, **10**	1, **8**, 10	4, **6**, 8	4, 8, **9**	**1**, 6, 9
R_4	2,5,6,7,8,10	2, **7**, 10	2, **6**, 10	5, 8, **10**	6, 7, **8**	**2**, 5, 8	**5**, 6, 7
R_5	3,4,5,7,9,10	4, 7, **10**	3, **9**, 10	3, **5**, 10	**3**, 4, 7	**4**, 5, 9	5, **7**, 9

- • *Assignment problem*: find $a_{ij} \in R_i \cap C_j$: $a_{ij} \neq a_{kj}$, $a_{ij} \neq a_{il}$
- ◇ NP-complete if R_i, C_j are arbitrary **Fon-Der-Flaass, 1997**

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		C_1 1,2,4,7,10	C_2 2,3,6,9,10	C_3 1,3,5,8,10	C_4 3,4,6,7,8	C_5 2,4,5,8,9	C_6 1,5,6,7,9
R_1	1,2,3,4,5,6	**1**, 2, 4	**2**, 3, 6	1, **3**, 5	3, **4**, 6	2, 4, **5**	1, 5, **6**
R_2	1,2,3,7,8,9	1, **2**, 7	2, **3**, 9	**1**, 3, 8	3, **7**, 8	2, **8**, 9	1, 7, **9**
R_3	1,4,6,8,9,10	1, **4**, 10	6, 9, **10**	1, **8**, 10	4, **6**, 8	4, 8, **9**	**1**, 6, 9
R_4	2,5,6,7,8,10	2, **7**, 10	2, **6**, 10	5, 8, **10**	6, 7, **8**	**2**, 5, 8	**5**, 6, 7
R_5	3,4,5,7,9,10	4, 7, **10**	3, **9**, 10	3, **5**, 10	**3**, 4, 7	**4**, 5, 9	5, **7**, 9

- *Assignment problem*: find $a_{ij} \in R_i \cap C_j$: $a_{ij} \neq a_{kj}$, $a_{ij} \neq a_{il}$
- ◇ NP-complete if R_i, C_j are arbitrary **Fon-Der-Flaass, 1997**
- ◇ Every extremal TA can be constructed this way:

  extremal TA $\rightarrow$ symmetric 2-design **MPWY, 2005**

Other constructions

- Constructions of extremal TA avoiding the assignment problem step
  - from *Hadamard matrices* **Preece–Wallis–Yucas, 2005**
  - from *Youden rectangles* **Nilson–Öhman, 2014**
  - from *difference sets* **Nilson–Cameron, 2017**

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- What about constructions of non-extremal triple arrays?
  - "Small" non-extremal admissible $(r \times c, v)$: $(7 \times 15, 35)$, $(11 \times 45, 99)$, $(15 \times 21, 63)$, $(16 \times 21, 56)$, $(16 \times 25, 100)$, $(13 \times 40, 130)$

◇ Is there a $(7 \times 15, 35)$-triple array? — **Preece, 1970s**

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◇ There is! **MPWY, 2005; Yucas, 2002**

31	1	18	16	7	10	5	3	4	2	33	14	19	15	12
26	32	1	2	29	30	28	20	27	11	5	34	3	8	4
1	17	13	9	3	4	21	22	6	35	25	5	24	2	23
6	27	33	28	16	13	35	30	15	10	9	26	12	17	29
16	12	23	32	34	21	15	33	24	22	11	10	8	25	20
21	22	28	24	25	19	7	14	18	29	27	23	26	30	31
11	7	8	14	13	32	20	6	34	18	19	17	35	31	9

◇ This is the only example of a non-extremal triple array known so far!

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26	32	**1**	**2**	29	30	28	20	27	11	**5**	34	**3**	8	**4**
1	17	13	9	**3**	**4**	21	22	6	35	25	**5**	24	**2**	23
6	27	33	28	16	13	35	30	15	10	9	26	12	17	29
16	12	23	32	34	21	15	33	24	22	11	10	8	25	20
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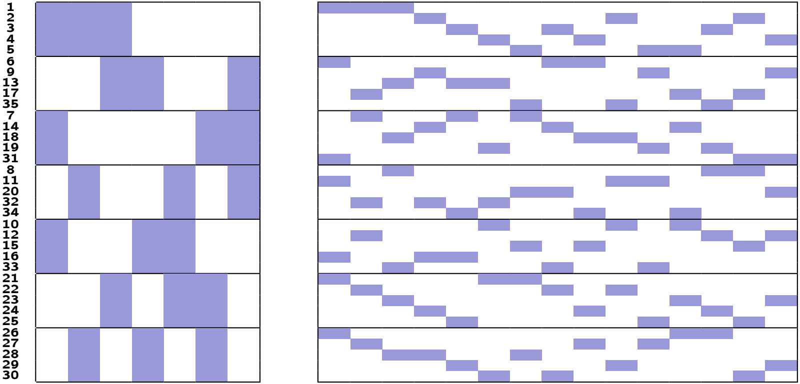
A non-extremal $(7 \times 15, 35)$-triple array



Row design

Column design

- Row design = 5 copies of a symmetric 2-design
- Column design = 7 *parallel classes* = *resolution* of a 2-design

◇ Can we get other triple arrays with a similar structure?

New triple array construction G.–Öhman, 2023+

- admissible $(r \times c, v)$, $a := \frac{e(e-1)}{r-1}$ and $k := \frac{c}{e}$ are integers

(1) choose a symmetric 2-(r, e, a) design (row design $= k$ copies of it)
(2) choose a resolution of a resolvable 2-(c, e, λ_{cc}) design (column design)
(3) choose a bijection between blocks of (1) and parallel classes of (2)
(4) solve the assignment problem

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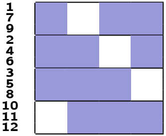
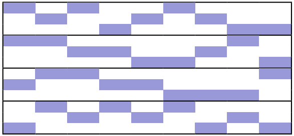
- ◇ First general construction for non-extremal TA
- ◇ In a constructed array every triple of two rows and a column have a common symbols
- ◇ Some extremal triple arrays can be constructed this way:

1	2	3	4	5	6	7	8	9
2	3	4	5	6	10	8	11	12
5	7	1	10	11	8	12	9	3
12	10	11	9	7	1	4	2	6





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New $(7 \times 15, 35)$-triple arrays

- 7 non-isomorphic resolutions of a $2\text{-}(15, 3, 1)$ design *(Kirkman parades)*
- Exhaustive search on a computer
- ◇ There are 85 pairwise non-isotopic $(7 \times 15, 35)$-TA with the described structure **G.–Öhman, 2023+**

# of parade	1	2	3	4	5	6	7
$\lvert\mathrm{Auth}\rvert$	168	168	24	24	12	12	21
TA found	0	3	24	4	21	21	12

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First $(21 \times 15, 63)$-triple arrays

- 149+ resolutions of a 2-$(15, 5, 6)$ design **Mathon–Rosa, 1989**
- Exhaustive search is out of the question
- Randomization + greedy nonexhaustive techniques

An example of a $(21 \times 15, 63)$-triple array

2	25	46	19	54	4	37	23	55	8	31	58	15	12	18	36	41	44	28	51	62
45	35	41	1	33	6	27	20	58	9	28	62	13	10	55	38	16	52	24	46	50
15	1	42	30	55	54	5	31	35	11	7	39	58	43	16	47	50	19	22	26	63
14	24	61	3	32	16	10	5	40	51	9	43	59	36	26	19	37	53	55	48	28
6	34	10	17	1	8	29	21	42	55	14	37	60	31	63	46	49	43	23	27	53
13	9	47	58	2	5	28	63	34	12	42	44	53	17	25	39	24	20	56	32	49
1	36	63	16	52	7	11	33	4	20	29	59	23	44	56	14	38	42	48	25	51
51	22	62	2	56	47	30	6	36	19	8	38	11	32	17	13	40	27	43	60	52
50	27	3	29	37	18	6	32	59	56	41	7	10	34	15	20	22	54	44	47	61
49	2	11	59	38	46	4	24	56	21	40	9	54	35	61	15	17	26	45	31	29
44	26	1	21	39	53	12	61	6	57	33	8	22	18	13	48	51	40	29	58	35
5	7	2	60	53	48	38	19	41	49	15	61	12	16	27	34	23	45	57	33	30
43	23	40	20	3	9	39	4	57	10	30	60	14	33	62	35	18	25	47	49	54
3	8	48	18	31	52	25	62	5	50	13	45	24	11	57	37	42	21	30	59	34
4	3	12	28	57	17	26	22	60	7	32	63	52	45	14	21	39	41	46	50	36

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45	35	41	**1**	33	6	27	20	58	9	28	62	13	10	55	38	16	52	24	46	50
15	**1**	42	30	55	54	5	31	35	11	7	39	58	43	16	47	50	19	22	26	63
14	24	61	**3**	32	16	10	5	40	51	9	43	59	36	26	19	37	53	55	48	28
6	34	10	17	**1**	8	29	21	42	55	14	37	60	31	63	46	49	43	23	27	53
13	9	47	58	**2**	5	28	63	34	12	42	44	53	17	25	39	24	20	56	32	49
1	36	63	16	52	7	11	33	4	20	29	59	23	44	56	14	38	42	48	25	51
51	22	62	**2**	56	47	30	6	36	19	8	38	11	32	17	13	40	27	43	60	52
50	27	**3**	29	37	18	6	32	59	56	41	7	10	34	15	20	22	54	44	47	61
49	**2**	11	59	38	46	4	24	56	21	40	9	54	35	61	15	17	26	45	31	29
44	26	**1**	21	39	53	12	61	6	57	33	8	22	18	13	48	51	40	29	58	35
5	7	**2**	60	53	48	38	19	41	49	15	61	12	16	27	34	23	45	57	33	30
43	23	40	20	**3**	9	39	4	57	10	30	60	14	33	62	35	18	25	47	49	54
3	8	48	18	31	52	25	62	5	50	13	45	24	11	57	37	42	21	30	59	34
4	**3**	12	28	57	17	26	22	60	7	32	63	52	45	14	21	39	41	46	50	36

Open questions

- Non-extremal triple arrays for other parameters
  - $\mathrm{PG}(2, q)$ + resolution of $\mathrm{PG}(3, q)$: $(7 \times 15, 35)$, $(13 \times 40, 130)$, $\ldots$
    - couldn't find a solution for $(13 \times 40, 130)$ even with nonexhaustive search
    - use "nice" resolutions? use geometry?

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    - ◇ use "nice" resolutions? use geometry?
  - $(11 \times 45, 99)$, $(15 \times 91, 195)$ $\ldots$: no resolvable 2-designs are known
  - $(16 \times 21, 56)$, $(16 \times 25, 100)$, $\ldots$: our construction is not applicable

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  - $(16 \times 21, 56)$, $(16 \times 25, 100)$, ...: our construction is not applicable
- Construction of non-extremal TA without the assignment problem step
  - ◇ Find sufficient conditions for our construction to work

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  - ◇ Find sufficient conditions for our construction to work
- **Agrawal's conjecture:** the assignment problem in Agrawal's construction has a solution if $\lambda_{rc} > 2$ (where $\lambda_{rc} = |R_i \cap C_j|$)

Open questions

- Non-extremal triple arrays for other parameters
  - $\mathrm{PG}(2,q)$ + resolution of $\mathrm{PG}(3,q)$: $(7\times 15, 35)$, $(13\times 40, 130)$, ...
    - ◇ couldn't find a solution for $(13\times 40, 130)$ even with nonexhaustive search
    - ◇ use "nice" resolutions? use geometry?
  - $(11\times 45, 99)$, $(15\times 91, 195)$ ...: no resolvable 2-designs are known
  - $(16\times 21, 56)$, $(16\times 25, 100)$, ...: our construction is not applicable
- Construction of non-extremal TA without the assignment problem step
  - ◇ Find sufficient conditions for our construction to work
- **Agrawal's conjecture:** the assignment problem in Agrawal's construction has a solution if $\lambda_{rc} > 2$ (where $\lambda_{rc} = |R_i \cap C_j|$)
- $v \geq r + c - 1$ in any $(r\times c, v)$-TA **Bagchi, 1998; MPWY, 2005**
  But are there even admissible $(r\times c, v)$ with $v < r + c - 1$?
  - ◇ Are there $r, c, v \in \mathbb{N}$, $\max(r,c) < v < r + c - 1$ such that

$$\frac{rc}{v} \in \mathbb{N},\ \frac{c(e-1)}{r-1} \in \mathbb{N},\ \frac{r(e-1)}{c-1} \in \mathbb{N}?$$